One-to-One Functions; Inverse Functions

One-to-one Function*:** A function is ***one-to-one if for any value of x there is exactly one y (otherwise it wouldn't be a function), and for any value of y, there is exactly one x.

Example: Determine whether the following functions are one to one.

a) $\{(-2,6),(-1,3),(0,2),(1,5),(2,8)\}$

one-to-one

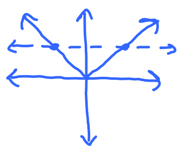
b) $\{(1,1),(2,4),(3,9),(0,0),(-1,1),(-2,4)\}$

no, 1 & 4 come from two different inputs

Horizontal Line Test: If every horizontal line intersects the graph of a function f in at most one point, then f is one-to-one.

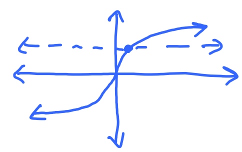
Example: For each function, use its graph to determine whether the function is one-to-one.

a) $f(x)=|x|$



Not one-to-one

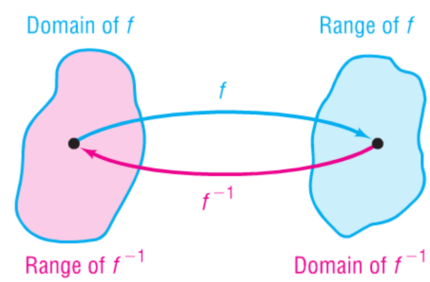
b) $g(x)=\sqrt[3]{x}$



one-to-one

Theorem: A function that is increasing on an interval I is a one-to-one function on I.
A function that is decreasing on an interval I is a one-to-one function on I.

Inverse Function*:** Two functions are ***inverses if and only if whenever one function contains the element (a,b), the other function contains the element (b,a). If f is a one-to-one function, the correspondence from the range of f back to the domain of f is called the ***inverse function*** of f. The inverse of f is abbreviated f^{-1}.



★ **Domain of f = Range of f^{-1}** **Range of f = Domain of f^{-1}**

Example: Find the inverse of the following one-to-one function: $\{(2,3),(4,5),(6,8),(9,10),(12,14)\}$

$\{(3,2),(5,4),(8,6),(10,9),(14,12)\}$

If we start with x, apply f, and then apply f^{-1}, we get x back again.
If we start with x, apply f^{-1}, and then apply f, we get x back again.
What f does, f^{-1} undoes, and vice versa. In other words,

$f^{-1}(f(x))=x$, where x is in the domain of f.

$f(f^{-1}(x))=x$ where x is in the domain of f^{-1}.

To verify that two functions are inverses, show that $f^{-1}(f(x))=x$ and $f(f^{-1}(x))=x$

Example: Verify that the inverse of $f(x)=\dfrac{2}{x+5}$ is $f^{-1}(x)=\dfrac{2}{x}-5$.

$f(f^{-1}(x)) = \dfrac{2}{\frac{2}{x}-5+5}$

$= \dfrac{2}{\frac{2}{x}}$

$= 2\left(\dfrac{x}{2}\right) = x$

$f^{-1}(f(x)) = \dfrac{2}{\frac{2}{x+5}} - 5$

$= 2\left(\dfrac{x+5}{2}\right) - 5$

$= x+5-5 = x$

Example: Verify that the inverse of $f(x)=\sqrt[3]{2x}$ is $f^{-1}(x)=\dfrac{x^3}{2}$.

$f(f^{-1}(x)) = \sqrt[3]{2\left(\frac{x^3}{2}\right)}$

$= \sqrt[3]{x^3}$

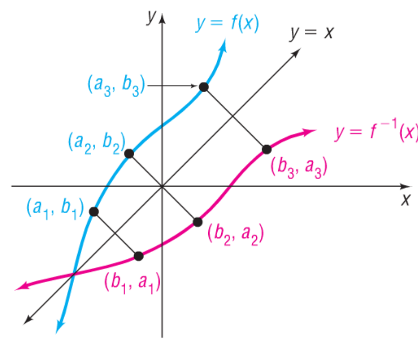
$= x$

$f^{-1}(f(x)) = \dfrac{\left(\sqrt[3]{2x}\right)^3}{2}$

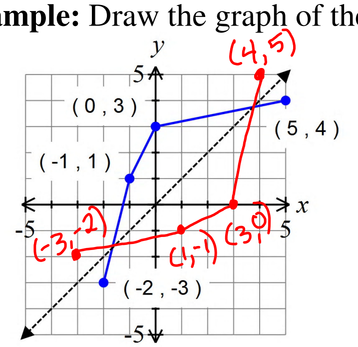
$= \dfrac{2x}{2}$

$= x$

Theorem: The graph of a function f and the graph of its inverse f^{-1} are symmetric with respect to the line $y=x$.



Example: Draw the graph of the inverse function.



Finding the Inverse of a Function

1. Rewrite $f(x)$ as y in the original equation.
2. Interchange x and y.
3. Solve for y.
4. Replace y with the notation $f^{-1}(x)$.

Example: Find the inverse. State the domain and range of $f(x)$ and the domain and range of $f^{-1}(x)$.

a) $f(x)=-3x+1$

$x = -3y+1$

$x-1 = -3y$

$y = \dfrac{x-1}{-3}$

$\boxed{f^{-1}(x) = \dfrac{x-1}{-3}}$

Domain of f: $(-\infty,\infty)$

Domain of f^{-1}: $(-\infty,\infty)$

Range of f: $(-\infty,\infty)$

Range of f^{-1}: $(-\infty,\infty)$

b) $f(x)=\dfrac{2x+3}{5x-4}$

$x = \dfrac{2y+3}{5y-4}$

$x(5y-4) = 2y+3$

$5xy-4x = 2y+3$

$5xy-2y = 4x+3$

$y(5x-2) = 4x+3$

$y = \dfrac{4x+3}{5x-2}$

$\boxed{f^{-1}(x) = \dfrac{4x+3}{5x-2}}$

Domain of f: $\{x \mid x \neq \frac{4}{5}\}$

Domain of f^{-1}: $\{x \mid x \neq \frac{2}{5}\}$

Range of f: $\{y \mid y \neq \frac{2}{5}\}$

Range of f^{-1}: $\{y \mid y \neq \frac{4}{5}\}$